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Decomposition Identities for Distinct Prime Partitions and Their Norms

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Abstract

In this paper, we study decomposition identities for partitions of positive integers into prime parts. In particular, we consider partitions into distinct primes. Let $D(n)$ denote the number of partitions of n into distinct prime parts and let $D_0(n)$ denote the number of partitions of n into distinct odd prime parts. By separating the distinct prime partitions according to the occurrence of the prime 2, we obtain the decomposition$D(n) = D_0(n) + D_0(n-2)$, with $D_0(n)$ for $n < 0$. We further introduce the prime-partition norm, defined as the product of the prime parts of a distinct prime partition, and derive decomposition and inverse formulas for the associated norm functions. The decomposition is subsequently extended from the prime 2 to an arbitrary prime. Finally, higher powers of the prime-partition norm are considered, leading to corresponding moment identities and generating functions. The results provide a unified decomposition framework for counting and weighted prime partitions and establish several relations between distinct prime partitions and their odd-prime counterparts.

1 Introduction

Partition theory is an important area of additive number theory concerned with representing positive integers as sums of positive integers. The theory of integer partitions is a classical topic in number theory, with extensive developments in generating functions, asymptotic formulas, and restricted partitions [1,2]. Among restricted partitions, prime partitions, in which the parts are required to be prime numbers, provide a natural connection between partition theory and the arithmetic properties of primes. Prime partitions, in which the parts are restricted to prime numbers, have also been studied extensively [3,5,7]. In this paper, we study partitions into distinct prime parts. Let $D(n)$ denote the number of partitions of n into distinct prime parts [2], and let $D_o(n)$ denote the number of partitions of n into distinct odd-prime parts. Since 2 is the only even prime, every distinct prime partition of n either contains 2 or does not contain 2. The partitions not containing 2 are precisely the distinct odd-prime partitions of n, while removing 2 from a partition containing 2 gives a distinct odd-prime partition of $n-2$. Consequently, $D(n) = D_0(n) + D_0(n-2)$, where $D_o(n) = 0$ for $n < 0$. This identity also follows directly from the generating function by separating the factor corresponding to the prime 2. Thus, the decomposition according to the smallest prime provides a common framework for distinct prime partitions. We

further introduce a weighted version of the distinct prime-partition function. For a distinct prime partition $\lambda = (p_1, p_2, \ldots p_k)$, its prime-partition norm is defined by $N(\lambda) = \prod_{i=1}^{k} p_i$.

By separating the partitions according to the occurrence of 2, we obtain a corresponding decomposition for the total norm. If $N_D(n)$ denotes the sum of the norms of all distinct prime partitions of n, and $N_0(n)$ denotes the corresponding sum over distinct odd-prime partitions, then $N_D(n) = N_0(n) + 2N_0(n-2)$. The factor 2 arises because restoring the removed prime 2 doubles the norm of the corresponding odd-prime partition. The norm decomposition also leads to an inverse relation. The decomposition can further be extended from the prime 2 to an arbitrary prime p, by separating distinct prime partitions according to whether p occurs as a part. This shows that the decomposition principle is not restricted to the prime 2. Finally, we consider higher powers of the prime-partition norm. For $s \geq 1$, define the s-th norm moment by $N_D^{(s)}(n) = \sum_{\lambda \in D(n)} N$ prime partitions λ of n. If a selected prime p occurs in a partition, restoring this part multiplies the k-th power of the norm by p^s. Hence, the same decomposition argument yields corresponding identities for higher norm moments and their generating functions. Thus, the results establish a unified decomposition framework for counting and weighted prime partitions and their corresponding odd-prime partition functions.

2 Generating Functions and Fundamental Decomposition Identities

Definition 2.1

Let $D(n)$ denote the number of distinct prime partitions of n. Its generating function is defined by $D(q) = \sum_{n=0}^{\infty} D(n)q^n$ [6,7]. Similarly, let $D_0(q) = \sum_{n=0}^{\infty} D_0(n)q^n$.

where $P = \{2,3,5,7, \ldots\}$ denote the set of all prime numbers [6,7].

Lemma 2.1.

The generating functions satisfy $D(q) = (1 + q^2)D_0(q)$. Equivalently, $\sum_{n=0}^{\infty} D(n)q^n = (1 + q^2)\sum_{n=0}^{\infty} D_0(n)q^n$.

Proposition 2.1. Generating Function for Distinct Prime Partitions

The generating function for the distinct prime partition function is $\sum_{n=0}^{\infty} D(n)q^n = (1 + q^2)\prod_{p>2}(1 + q^p)$.

Proof:

We know that $D(q) = \prod_{p \in P}(1 + q^p)$. By definition 2.1

Separating the factor $\prod_{p>2}(1 + q^p) = D_0(q)$.

corresponding to the prime

Therefore, $D(q) = (1 + q^2)D_0(q)$.

Theorem 2.1

For every integer $n \geq 2$, $D(n) = D_0(n) + D_0(n-2)$. Where $D_0(n) = 0$ whenever $n < 0$.

Proof:

Using lemma 2.1, $D(q) = (1 + q^2)D_0(q)$,

$$\sum_{n=0}^{\infty} D(n)q^n = \sum_{n=0}^{\infty} D_0(n)q^n + q^2 \sum_{n=0}^{\infty} D_0(n)q^n.$$

We obtain

Since $q^2 \sum_{n=0}^{\infty} D_0(n)q^n = \sum_{n=2}^{\infty} D_0(n-2)q^n$,

It follows that $\sum_{n=0}^{\infty} D(n)q^n = \sum_{n=0}^{\infty} (D_0(n)+D_0(n-2))q^n$.

Comparing the co-efficient of q^n on both sides gives $D(n) = D_0(n) + D_0(n-2)$.

Numerical Verification:

Example 2.1

Distinct prime partitions of 12 are 12 = 5+7 ; 12 = 2+3+7.

$D(12) = 2$.

Since $D_0(12) = 1$ and $D_0(10) = 1$.

We obtain $D(12) = D_0(12) + D_0(10) = 1 + 1 = 2$.

Corollary 2.1 (Even Prime Partition Identity)

Let $EPP(n)$ denote the number of distinct prime partitions of n that contain the even prime 2, and let $OPP(n)$ denote the number of distinct odd-prime partitions of n. Then, for every integer $n \geq 2$, $EPP(n) = OPP(n-2)$.

Theorem 2.2:

Let $D(n)$ denote the number of partitions of n into distinct prime parts, and let $D_0(n)$ denote the number of partitions of n into distinct odd prime parts. Then, for every integer $n \geq 0$,

$D_0(n) = \sum_{k=0}^{\frac{n}{2}} (-1)^k D(n-2k)$, where $DP(m) = 0$ whenever $m < 0$.

Proof:

From the decomposition theorem, $D(n) = D_0(n) + D_0(n-2)$----------(1)

Rearranging (1), $D_0(n) = D(n) - D_0(n-2)$.----------(2)

Applying (2) again to $D_0(n-2)$,

$D_0(n-2) = D(n-2) - D_0(n-4)$.

Substituting into (2), $D_0(n) = D(n) - (D(n-2) - D_0(n-4))$

$= D(n) - D(n-2) + D(n-4) - D(n-6) + D_0(n-8)$.

Continuing in this way, the process terminates because eventually the argument becomes negative, and by convention $D_0(m) = 0,\ m < 0$.

Therefore, $D_0(n) = D(n) - D(n-2) + D(n-4) - D(n-6) + \cdots$,

or equivalently,

$$D_0(n) = \sum_{k=0}^{\frac{n}{2}} (-1)^k D(n-2k)$$

3. Prime-Partition Norm and Its Decomposition

Definition: (Prime Partition Norm)
Let $\lambda = (p_1, p_2, \ldots p_k)$ be a distinct prime partition of n. The prime partition norm of λ is defined by $N(\lambda) = \prod_{i=1}^{k} p_i$. That is, the norm is simply the product of all prime parts [1,2].
Example:
The distinct prime partitions of 10 are 10 = 3+7, and 10 = 2+3+5.
Their norms are $N(3+7) = 21$. and $N(2+3+5) = 30$.
Hence $N(10) = 21 + 30 = 51$.
Definition: (Odd Prime Partition Norm)
$N(n) = \sum_{\lambda \in D(n)} N(\lambda)$, where the sum extends over all distinct prime partitions of n. Similarly,
$N_0(n) = \sum_{\lambda \in D_0(n)} N(\lambda)$, denotes the sum of norms over all distinct odd prime partitions of n.

Example:

For $n = 10$.

$N(10) = 21 + 30 = 51.$

Since $D_0(10) = 3 + 7$

We have, $N_0(10) = 21.$

Theorem 3.1:

For every integer $n \geq 2$, $N(n) = N_0(n) + 2N_0(n-2)$.

Proof:

The set of distinct prime partitions of n into two disjoint classes.

Case 1.

The partition does not contain the prime 2. Then every pair is an odd prime. Hence the contribution of these partitions is $N_0(n)$.

Case 2.

The partitions contain the prime 2. Suppose $\lambda = 2 + p_1 + p_2 + \cdots + p_k$,

Its norm equals $N(\lambda) = 2p_1p_2 \ldots p_k$.

Removing the prime 2 produces the distinct odd prime partition $p_1 + p_2 + \cdots + p_k$ of

$n - 2$,.whose norm is $p_1, p_2, \ldots p_k$.

Therefore $N(\lambda) = 2N(\lambda')$, where $\lambda' = p_1 + p_2 + \cdots + p_k$.

Summing over all partitions containing 2,

$\sum_{\lambda \in 2} N(\lambda) = 2N_0(n-2)$.

Combining both classes, $N(n) = N_0(n) + 2N_0(n-2)$.

Corollary 3.1

The generating functions satisfy,

$$\sum_{n=0}^{\infty} N(n)\, q^n = (1 + 2q^2) \sum_{n=0}^{\infty} N_0(n)\, q^n.$$

Proof:

Multiply the recurrence $N(n) = N_0(n) + 2N_0(n-2)$, by q^n and sum over all $n \geq 0$.

$\sum_{n\geq 0} N(n)q^n = \sum_{n\geq 0} N_0(n)q^n + 2\sum_{n\geq 0} N_0(n-2)q^n$.

Since $\sum_{n\geq 2} N_0(n-2)q^n = q^2 \sum_{n\geq 0} N_0(n)q^n$,

We obtain $\sum_{n\geq 0} N(n)q^n = (1+2q^2)\sum_{n\geq 0} N_0(n)q^n$.

Numerical verification

Example

Take $n = 10$.

Distinct prime partitions 3+7,2+3+5. Thus $N(10) = 21 + 30 = 51$

Now $N_0(10) = 21, N_0(8) = 15$.

Because 8 = 3+5,

Therefore $N_0(10) + 2N_0(8) = 21 + 2(15) = 51 = N(10)$.

Applications of the Prime-Norm Decomposition

Norm of partitions containing the prime 2
Let $N_D(n)$ denotes the sum of the norms of all distinct prime partitions of n, while $N_0(n)$ denotes the sum of the norms of all distinct odd-prime partitions of n.
Using the above theorem, gives $N_D(n) = N_0(n) + 2N_0(n-2)$.
Therefore, the total norm contributed by those distinct prime partitions of n that contain 2 is $N_E(n) = 2N_0(n-2)$, where $N_E(n)$ denotes the norm contribution from partitions the prime 2.

Example

For $n = 12$,

12 = 5+7, 12 = 2+3+7.

Their norms are 35, 42.

Thus $N_D(n) = 35 + 42 = 77$.

The partition containing 2 has norm 42.
Removing 2 gives 10 = 3+7, whose norm is 21.

Hence $N_E(12) = 2N_0(10) = 2(21)$=42. Thus the decomposition is 77 = 35+42. An inverse formula for odd-prime norms

The above theorem can be rearranged as $N_0(n) = N_D(n) - 2N_0(n-2)$.

Applying the same identity to $N_0(n-2)$,

$N_0(n-2) = N_D(n-2) - 2N_0(n-4)$.

Therefore, $N_0(n) = N_D(n) - 2N_D(n-2) + 4N_0(n-4)$.

Continuing recursively gives $N_0(n) = N_D(n) - 2N_D(n-2) + 4N_D(n-4) - 8N_D(n-6) + \cdots$,

Theorem 3.2: Inverse Norm Formula

For every $n \geq 0$, $N_0(n) = \sum_{j=0}^{n/2} (-2)^j N_D(n-2j)$.

Proof:

Using the theorem, $N_D(n) = N_0(n) + 2N_0(n-2)$.

$N_0(n) = N_D(n) - 2N_0(n-2)$.

Applying the same identity to $N_0(n-2)$,

$N_0(n-2) = N_D(n-2) - 2N_0(n-4)$.

Therefore, $N_0(n) = N_D(n) - 2N_D(n-2) + 4N_0(n-4)$.

Continuing recursively gives $N_0(n) = N_D(n) - 2N_D(n-2) + 4N_D(n-4) - 8N_D(n-6) + \cdots$,

Repeating this process k times gives

$$N_0(n) = \sum_{j=0}^{k} (-2)^j N_D(n-2j) + (-2)^{k+1} N_0(n-2k-2).$$

When $k = \frac{n}{2}$, the remaining argument is negative or zero, and under the convention $N_0(m) = 0$ for $m < 0$, the process terminates. Therefore, $N_0(n) = \sum_{j=0}^{\frac{n}{2}} (-2)^j N_D(n-2j)$.

Example : Take $n = 10$.

From the norm values, $N_D(10) = 51; N_D(8) = 15, N_D(6) = 0, N_D(4) = 0, N_D(2) = 2$.

Then $N_0(10) = N_D(10) - 2N_D(8) + 4N_D(6) - 8N_D(4) + 16N_D(2) - 32N_D(0)$

$= 51 - 2(15) + 4(0) - 8(0) + 16(2) - 32.$

This gives 51 – 30 + 32 -32 =21.

4. Application to arbitrary primes

Let r be any prime and let $N_r(n)$ denote the total norm of distinct prime partitions of n that do not r.

Then $N_D(n) = N_r(n) + rN_r(n - r)$.

For $r = 2$, $N_D(n) = N_0(n) + 2N_0(n - 2)$. (taking $r = 2$, the partitions avoiding 2 are precisely the distinct odd-prime partitions of n).

For $r = 3$, $N_D(n) = N_3(n) + 3N_3(n - 3)$.

For $r = 5$, $N_D(n) = N_5(n) + 5N_5(n - 5)$.

Thus, the decomposition can be performed with respect to any chosen prime, not only 2.

Higher Norm Moments

The preceding theorem concerns the first power of the prime-partition norm. A natural extension is to consider higher powers of the norm.

For $s \geq 1$, define the $s - th$ norm moment by $N_D^{(s)}(n) = \sum_{\lambda \in D(n)}$

Similarly, define $N_r^{(s)}(n) = \sum_{\lambda \in D_r(n)} N(\lambda)^s$.

Clearly, $N_D^{(1)}(n) = N_D(n)$.

For $r = 2$, define $N_0^{(s)}(n) = N_2^{(s)}(n)$.

Theorem:4.1

For every prime r, every integer $s \geq 1$, and every $n \geq 0$, $N_D^{(s)}(n) = N_r^{(s)}(n) + r^s N_r^{(s)}(n - r)$.

Proof:

Divide the distinct prime partitions of n into two classes.

The partitions not containing r contribute $N_r^{(s)}(n)$.

For a partition containing r, $\lambda = (r, p_1, p_2, \ldots p_k)$,

Remove r to obtain $\lambda' = (p_1, p_2, \ldots p_k)$,

Since $N(\lambda) = rN(\lambda')$,

We have $N(\lambda)^s = (rN(\lambda'))^s = r^s N(\lambda')^s$.

Therefore the total contribution from all partitions containing r is $r^s N_r^{(s)}(n-r)$.

Hence $N_D^{(s)}(n) = N_r^{(s)}(n) + r^s N_r^{(s)}(n-r)$.

Corollary4.1:

For $r = 2$, $N_D^{(s)}(n) = N_0^{(s)}(n) + 2^s N_0^{(s)}(n-2)$.

For $s = 1$, $N_D(n) = N_0(n) + 2N_0(n-2)$.

For $= 2$, $N_D^{(2)}(n) = N_0^{(2)}(n) + 4N_0^{(2)}(n-2)$.

For $s = 3$, $N_D^{(3)}(n) = N_0^{(3)}(n) + 8N_0^{(3)}(n-2)$.

Example:

For n =10, the distinct prime partitions are 10 = 3+7, 2+3+5.

Their norms are 21, 30.

Therefore $N_D^{(2)}(10) = 21^2 + 30^2 = 441 + 900 = 1341$.

The distinct odd-prime partitions of 10 consist of 3+7,

So $N_0^{(2)}(10) = 21^2 = 441$.

For $n - 2 = 8$, the distinct odd-prime partition is 3+5, with norm 15.

Hence $N_0^{(2)}(8) = 15^2 = 225$.

$$N_D^{(2)}(10) = 441 + 4(225)$$
$$= 441 + 900$$
$$= 1341$$

Therefore the identity is verified.

5. Generating Functions for Norms and Norm Moments

Define $N_D^{(s)}(q) = \sum_{n\geq 0} N_D^{(s)}(n)q^n$ and $N_r^{(s)}(q) = \sum_{n\geq 0} N_r^{(s)}(n)q^n$.

Theorem5.1:

For every prime r and integer ≥ 1, $N_D^{(s)}(q) = (1 + r^s q^r) N_r^{(s)}(q)$.

Proof: $N_D^{(s)}(n) = N_r^{(s)}(n) + r^s N_r^{(s)}(n - r)$.

Multiply both sides by q^n and sum over n ≥ 0.

$$\sum_{n\geq 0} N_D^{(s)}(n) q^n = \sum_{n\geq 0} N_r^{(s)}(n) q^n + r^s \sum_{n\geq 0} N_r^{(s)}(n - r) q^n.$$

The first sum is $N_r^{(s)}(q)$.

For the second sum, put $m = n - r$.

Then $\sum_{n\geq 0} N_r^{(s)}(n - r) q^n = q^r \sum_{m\geq 0} N_r^{(s)}(m) q^m$.

Therefore, $N_D^{(s)}(q) = N_r^{(s)}(q) + r^s q^r N_r^{(s)}(q)$.

Hence $N_D^{(s)}(q) = (1 + r^s q^r) N_r^{(s)}(q)$.

Corollary 5.1:

$r = 2$, we obtain $N_D^{(s)}(q) = (1 + 2^s q^2) N_0^{(s)}(q)$. Since $N_2^{(s)}(q). = N_0^{(s)}(q)$.

For $s = 1$, $N_D(q) = (1 + 2q^2) N_0(q)$.

For $s = 2$, $N_D^{(2)}(q) = (1 + 4q^2) N_0^{(2)}(q)$.

For $s = 3$, $N_D^{(3)}(q) = (1 + 8q^2) N_0^{(3)}(q)$.

6 Conclusion

In this paper, we have developed a decomposition framework for prime partitions, with particular emphasis on partitions into distinct prime parts. By separating distinct prime partitions according to the occurrence of the prime 2, we obtained the fundamental relation between the number of distinct prime partitions and the number of distinct odd-prime partitions. A similar decomposition was established for prime partitions in which repetition is allowed by separating according to the number of occurrences of 2. We then introduced the prime-partition norm, defined as the product of the prime parts of a distinct prime partition, and established the corresponding norm decomposition. The resulting identity yields an inverse formula for the odd-prime norm function in terms of the total prime-partition norm. These results demonstrate that the decomposition principle can be extended from counting functions to weighted partition functions. Furthermore, the decomposition was generalized with respect to an arbitrary prime p, showing that the role of the prime 2 is a particular case of a more general prime-partition decomposition. We also introduced higher powers of the prime-partition norm and derived corresponding identities for

higher norm moments. The associated generating functions provide an algebraic formulation of these decomposition properties. The results obtained in this work establish a unified connection between distinct prime partitions, odd-prime partitions, and their associated norm functions. They also provide a framework for further investigations involving other arithmetic weights, higher-order norm statistics, and related restricted prime-partition functions.

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