The Bioscan
AN INTERNATIONAL QUARTERLY JOURNAL OF LIFE SCIENCES
19(2): S.I (1), 9-18, 2024
www.thebioscan.com

$\sigma - p.i.c. -$ functors in the category of $Tych$ Tikhonov spaces and continuous mappings into themselves and dimension dim.

Gulchekhra Mukhamedova,

Candidate of pedagogical sciences, docent of the department of "General Mathematics" of Tashkent State Pedagogical University named after Nizami.

e-mail: muxamedovagulchehra74@gmail.com, https://orcid.org/ 0000-0002-1252-5790.

Feruza Saydalieva,

Candidate of pedagogical sciences, docent of the department of "General Mathematics" of Tashkent State Pedagogical University named after Nizami.

e-mail: fsaidaliyeva@gmail.com

Dilnoza Ibragimova,

assistant of the department of "General Mathematics" of Tashkent State Pedagogical University named after Nizami.

e-mail: ibragimovadilnoza1983@gmail.com

Kamariddin Zhuvonov,

Assistant of the department of "Higher mathematics" Tashkent Institute of Irrigation and Agricultural Mechanization Engineers, National Research University.

e-mail: qamariddin.j@mail.ru,

Tursunboy Zhuraev,

Doctor of Physical and Mathematical Sciences, Professor of the "Department of General Mathematics" at Tashkent State Pedagogical University named after Nizami.

e-mail: tursunzhuraev@mail.ru, https://orcid.org/0009-0005-5379-3862

DOI: https://doi.org/10.63001/tbs.2024.v19.i02.S.I(1).pp09-18

KEYWORDS
Normal functor, locally convex subfunctor, weakly countable dimension, finitely open functor, projectively inductively closed functor, functor, finite functor.

Received on: 25-07-2024

Accepted on: 09-11-2024

ABSTRACT

This note studies the geometric, topological and dimensional properties of the $\sigma - p.i.c$ -functor and its subfunctors in the category of $Tych -$ Tychonov spaces and continuous self-maps. It is shown that $\sigma - p.i.c$. Functors preserve weakly countable-dimensional spaces, $\sum$ –paracompact, $\sigma -$ paracompact, $p -$ paracompact, metrizable and stratifiable spaces.

Considering a number of subfunctors F of the functor P of probability measures that are $\sigma - p.i.c$.– functors, their various topological and dimensional properties are studied, properties in the category of $Top -$ topological spaces and continuous mappings into themselves.

MSC. 54B15, 54B30, 54B35, 54C05, 54C60, 54O30.

INTRODUCTION

For metric compact sets X there is a simple topological classification of the spaces $P(X)$ of all probability measures. In the case of a finite $n-$point space $X=\{n\}$ the points μ of the space $P(n)=P_n(n)$ are convex linear combinations of Dirac measures:

$$\mu=m_0\delta(0)+m_1\delta(1)+...+m_{n-1}\delta(n-1)$$

Therefore, they are naturally identified with the points of the $(n-1)$ dimensional simplex σ^{n-1}. In this case, Dirac $\delta(i)$ are formed by the vertices of the simplex, and the masses m_i placed at points i are the barycentric coordinates of the measure μ. Thus, the compact set $P(n)$ is affinely homeomorphic to the simplex σ^{n-1} [1-2].

In the case of an infinite compact space X, the space

It is also known that any χ_1 -degree of a non-one-point compact set K, the space of all probability measures $P(K^{\chi_1})$ is homeomorphic to the Tikhonov cube I^{χ_1}, $P(K^{\chi_1})=I^{\chi_1}$, $I-$ segment $[0,1]$. Note, in particular, that all these spaces are topologically homogeneous. A for spaces $P(K^{\chi_1})$ for $\tau>\chi_1$ the situation is different [3].

For an arbitrary compact set X and a measure $\mu\in P(X)$, its support $supp(\mu)$ is defined, which is

$$\mu=m_1\delta(x_1)+m_2\delta(x_2)+...+m_n\delta(x_n)$$

with finite supports, for each of which $m_i\geq\frac{n}{n+1}$ for some i;

$P_{f,n}(X)=\{\mu\in P_f(X):|supp\mu|\leq n\}$.

Hence, the compact subspace $P_f(X)$ is also the union of compacts $P_{f,n}(X)$. those. $P_f(X)=\bigcup_{n=1}^{\infty}P_{f,n}(X)$. It is obvious that $P_{f,n}(X)\subseteq P_n(X)$ and $P_f(X)\subseteq P_\omega(X)$.

Obviously, for a metric compact set X and any

$P(X)$ is also a compact space [3]. Further, it, containing simplices of an arbitrarily large number of dimensions, is infinite-dimensional [4]. By Cayley's theorem, the convex compact set $P(X)\subset R^{C(X)}$ is affinely embedded in ℓ_2 . Consequently, by Keller's theorem, the compact set $P(X)$, as an infinite-dimensional convex compact set lying in ℓ_2, is homeomorphic to the Hilbert cube $Q=I^{\chi_0}$ [2]. On the other hand, the space $P(X)$ of all probability measures on a compact set X is called the set of all regular Borel probability measures on X, equipped with the weakest of the topologies for which each functional $f_u:C(X)\rightarrow R$, which takes the measure μ to $\mu(U)$ (U is an open set in X) [5].

the smallest of the closed sets $F\subset X$ for which $\mu(F)=\mu(X)$ i.e.

$$\sup p(\mu)=\bigcap\{A:A=\overline{A},\mu\in P(A)\};$$

$$P_n(X)=\{\mu\in P(X):|supp\mu|\leq n\}-$$

the set of all measures μ at most n supported, $P_\omega(X)=\bigcup_{n=1}^{\infty}P_n(X)-$ the set of all probability measures μ with finite supports[4-5]. Recall that the space $P_f(X)\subset P(X)$ consists of all probability measures [1]

$n\in N$, the sets $P_n(X)$, $P_f(X)$, $P_{f,n}(X)$ are closed in $P(X)$.

Consequently, the subspace $P_\omega(X)\subset P(X)$ and $P_\omega(X)$ is $\sigma-$ compact, everywhere dense in $P(X)$.

Let $P^c(X)$ denote the set of all measures $\mu\in P(X)$, the support of each of which lies in one of the connected components of the space (compact space) X [2].

The closedness of the set $P^c(X)$ in the compact set $P(X)$ is quite obvious. The inclusion $P(f)(P^c(X))\subset P^c(Y)$ follows from the fact that the functor P, like any normal functor, preserves supports, i.e.

. $supp(P(f))(\mu_1)=f(supp\mu)$.

Thus, the subfunctor P^c of the functor P is defined. An epimorphism $f:C\to I$ of a Cantor perfect set onto an interval shows that the functor P^c does not clearly preserve epimorphisms in the same way that $P^c\neq P_1$, while as $P^c(n)=n=P_1(n)$. Therefore, we get $P_f^c\equiv P_f\bigcap P^c$, $P_{f,n}^c\equiv P_f\bigcap P^c\bigcap P_n$, $P_n^c\equiv P^c\bigcap P_n$, $P_\omega(X)\bigcap P^c=P_\omega^c$.

For an infinite compact set $X\in Comp$ and a normal or seminormal functor $F:Comp\to Comp$ of infinite degree [1]. Let us accept the following notation[6]:

1) $F_\nabla(X)=F(X)\setminus\eta_F(X)$; For $n=1$ we identify $F_\nabla(X)$; $F_{\nabla 1}(X)$;

2) $F_n(X)=\{a\in F(X):|supp(a)|\leq n\}$. $supp(a)$ denotes the support of the point a $a\in F(X)$;

3) $F_{\nabla n}(X)=F(X)\setminus F_n(X)$.

4) $S_F(A)=\{a\in F(X):supp(a)\bigcap A\neq\varnothing$, $A\neq\varnothing$ and $A\subset X\}$;

5) $F_{nk}(X)=F_n(X)\setminus F_k(X), n>k, n\geq 2$;

6) $F_\omega(X)=\bigcup_{n=1}^\infty F_n(X)$;

7) $F_{\nabla\omega}(X)=F(X)\setminus F_\omega(X)$;

8) $F_{\omega n}(X)=F_\omega(X)\setminus F_n(X)$;

Recall that the topological space *Y* is called an absolute (neighborhood) retract in the class *K* (written $Y\in A(N)R(K)$, if $Y\in K$ and for any homeomorphism *h*, mapping *Y* onto a closed subset *h*(*Y*) of the space *X* from the class *K*, the set *h*(*Y*) is a retract (neighborhood) of the space *X* [7].

Recall that a topological space X is called a manifold modeled on the space Y, or a $Y-$manifold [7], if every point in the space X has a neighborhood homeomorphic to an open subset of the space X.

A Q-manifold is a separable metric space locally homeomorphic to the Hilbert cube Q [7], where $Q=\prod_{i=1}^\infty[-1,1]_i$; Hilbert cube [8], $W_i^\pm=\{(g_j)\in Q\mid g_i=\pm1$ $j-$th face of the Hilbert cube Q, $BdQ=\bigcup_{i=1}^\infty W_i^\pm -$ is called the pseudo-boundary of the cube Q, and $S=Q\setminus BdQ-$ the pseudo-interior of the cube Q. It is known that $S=\prod_{i=1}^\infty(-1,1)_i$.

In the theory of infinite-dimensional manifolds, the following objects play an important role: the Hilbert cube Q, the separable Hilbert space ℓ_2, $\sum-$the linear hull of the standard brick $Q'=\prod_{n=1}^\infty[0,\frac{1}{2^n}]$ in the Hilbert space ℓ_2, via ℓ_2^f or σ denotes the linear subspace of the Hilbert space ℓ_2, consisting of all points only a finite number of coordinates of which is nonzero, and Q^f is the subspace of the Hilbert cube Q, consisting of only a finite number of points the number of coordinates of which is different from zero.

By the Anderson-Kadets theorem [9], the Hilbert space ℓ_2 is homeomorphic to *C* [9]. From Bessaga-Pelczynski's results it follows that $\sum$ is homeomorphic to $rintQ$ and σ; ℓ_2^f; Q^f [9]. Here $rintQ$ denotes the set $\{x=(x_n)\in Q\,|\,|x_n|<t<1$ for all $n\in N\}$. Further, it is obvious that $rintQ\approx BdQ$, which means it is true: $BdQ\approx\sum$.

It is known that the spaces Q, $\sum$ and ℓ_2 are strongly infinite-dimensional, and the spaces ℓ_2^f, σ and Q_f are weakly infinite-dimensional and all these spaces are homogeneous.

Definition [10]. A closed set *A* of space *X* is called Z-set in X if the identity mapping id_X of the space X can be approximated arbitrarily closely by the mappings $f:X\to X\setminus A$.

A countable union of Z-sets in *X* is called a $\sigma-Z-$set in X.

Following [9], a $\sigma-Z-$set B of a Hilbert cube Q is called a boundary set in Q (denoted by $B(Q)$) if $Q\setminus B\approx\ell_2$. More generally, a boundary set in a Q-manifold is a $\sigma-Z-$set whose complement is a ℓ_2-manifold.

From the above it follows that the pseudo-boundary

BdQ of the Hilbert cube Q is its boundary set.

Let X be a topological space.

A set $A \subset X$ is called homotopy dense in X [10] if there is a homotopy $h(x,t): X \times [0,1] \to X$ such that $h(x,0) = id_X$ and $h(X \times (0,1]) \subset A$.

An A set $A \subset X$ is homotopy negligible in X if $X \setminus A$ is homotopy dense in X.

An embedding $e: Y \to X$ is homotopy dense (resp. homotopy negligible) if $e(Y)$ is a homotopy dense set (resp. homotopy negligible) in X [5].

Definition 1. A covariant functor $F: Comp \to Comp$ is called normal [3] if the following conditions are met: continuous; monomorphic; epimorphic; saves: weight; intersections; prototypes; point and empty set;

Definition 2. The functor $F: Comp \to Comp$ is called [11] regular if the following are satisfied: monomorphic; epimorphic, continuous, preserving intersections and preimages.

Definition [2]. A normal subfunctor F of a functor P_n is said to be locally convex if the subset $F(n)$ of the simplex $P_n(n)$ is locally convex.

Let F and G be functors. A natural transformation $T: \mathrm{F} \to G$ between us is the system of mappings $T_X: \mathrm{F} \to G, T_Y: \mathrm{F} \to G$, where X is a space such that for any mapping $f: X \to Y$ between the spaces X and Y the following holds: $G(f) \circ T_X = T_Y \circ \mathrm{F}(f)$.

A functor F is called a subfunctor of a functor G if there is a natural transformation $T: G \to \mathrm{F}$ such that T_X embedding for any X.

Main part

For a continuous mapping $f: X \to Y$ between the Tikhonov spaces X and Y, a mapping $\beta f: \beta X \to \beta Y$ is defined, which satisfies the conditions

$$F(\beta f)(F(\beta X)) \subset F_\beta(Y) \tag{1}$$

$$f(supp(a)) \supset supp(F(f)(a)) \tag{2}$$

where $F_\beta(X) = \{a \in F(\beta X) : supp(a) \subset X\}$, βX is the Stone Chekhov extension of the space X.

For a functor F and an element $a \in F(X)$, the support of a point $supp(a)$ is the intersection of all closed sets of the space X such that $a \in F(A)$ i.e. $supp_F(a) = \bigcap \{A \subset X : A$ is closed and $a \in F(X)\}$

Definition 3. A functor $F: Tych \to Tych$ is called compact ($\sigma -$compact) if $F(K)$ is compact ($\sigma -$compact) for any space $X \in Comp$.

Comment 4. For any functor $F: Comp \to Comp$, the functor F_β is compact, but not every compact functor in $Tych$ is of type F_β. As examples, we can consider the functors P_R and P_τ of Radon and τ-additive probability measures, respectively (see [11]).

Returning to the $F: Comp \to Comp$ functors, we obviously have

$$a \in F(supp(a)). \tag{3}$$

If the functor F preserves preimages. then F preserves supports [11], i.e.

$$f(supp(a) = supp(F(f)(a)). \tag{4}$$

The (1.4) property can be converted.

Proposition 5 [1]. Any functor in $Comp$ is support-preserving if and only if it is preimage-preserving.

From the definition of the functor F_β and the property (4) it follows that

$$f(supp_{F_\beta(X)}(a) = supp_{F_\beta(Y)} F_\beta(f)(a) \tag{5}$$

for any preimage-preserving functor $F: Comp \to Comp$, Tikhonov spaces X and Y, continuous mapping $f: X \to Y$. and $a \in F_\beta(X)$.

For the category *C* we denote by *O*(*C*) and *M*(*C*) the family of all objects *C* and the family of all morphisms *C* respectively. Let *G* and *F* be functors in some category $C \subset Top$. A natural transformation $i: G \to F$ is a family of mappings $i_X: G(X) \to F(X)$, $X \in O(C)$, such that

$$F(f) \circ i_X = i_Y \circ G(f) \tag{6}$$

for any $f \in M(C)$.

The mappings i_X are called components of the transformation i. A functor G is called a subfunctor of a functor *F* if there is a natural transformation $i: G \to F$ such that all its components i_X are embeddings. This natural transformation of i is called the embedding of *G* into *F*. If $i: G \to F$ is an embedding, then (6) implies

$$G(f) = i_Y^{-1} \circ F(f) \circ i_X \quad (7)$$

Comment 7. Usually subfunctors arise in a fairly natural way. For example, in the category *Comp* the functor $\exp^c$ of subcontinuum hyperspaces is built into the functor *exp* of hyperspaces of closed subsets from the very beginning. Let's assume this is a standard situation. This means that the functor *G* is a subfunctor of *F* if the natural transformation (embedding) $i: G \to F$ consists of identical embeddings, i.e. *G*(*X*) is a subspace in *F*(*X*) for any space *X*. In this case, the condition (7) is equivalent to the condition

$$F(f)(G(X)) \subset G(Y). \quad (8)$$

Definition 8. Let F_k -subfunctor of the functor *F* in *Comp*, defined as follows. A-priory. $F_k(\varnothing) = F(\varnothing)$. For a non-empty compactum *X*, $F_k(X)$ is the image of the mapping $\pi_{F,X,k}$. For a mapping $f: X \to Y$, $F_k(f)$ is the restriction of $F(f)$ to $F_k(X)$. Let $\overline{f}: C(k,X) \to C(k,Y)$ denote the mapping that takes ξ onto the composition $f \circ \xi$. It's easy to see that

$$\pi_{Y,k} \circ \overline{f} \times id_{F(\{k\})} = F(f) \circ \pi_{X,k} \quad (9)$$

Therefore, $F(f)(F_k(X)) \subset F_k(Y)$. Therefore F_k is a functor. Clear. F_k is a subfunctor of *F* with unit embedding $F_k(X) \subset F(X)$ for an arbitrary compact set *X*.

A functor *F* is called a degree functor *k* (they write deg*F* = *k*) if $F_k(X) = F(X)$ for any compact set X, and $F_{k-1}(X) \neq F(X)$ for some *X*[3].

Proposition 9[11]. For any continuous functor *F* and compact set *X* we have

$$F_k(X) = \{a \in F(X) : |supp(a)| \le k\}.$$

The media definition and properties (3) imply

Proposition 10[13]. For a functor *F*, a compact space *X* and a closed subset A in *X* we have

$$F(A) = \{a \in F(X) : supp(a) \subset A\}.$$

Proposition 11[13]. Let *F* be a functor in *Tych*, A be a closed subset of *X*, $a \in F(X)$ and $supp(a) \subset A$. Then $F(f)(a) = a$ for any mapping $f: X \to X$ such that $f \mid A = id_A$.

Proposition 12[14]. Let *F* be a regular functor in *Comp*, and *G* its regular subfunctor. Then

$$G(A) = G(X) \bigcap F(A)$$

for any compact set *X* and its closed subset *A*.

From this definition of a carrier we get

Proposition 13[14]. Let *F* be a functor in *Comp*, and *G* its subfunctor. Then

$$supp_{F(X)}(a) \subset supp_{G(X)}(a)$$

for any compact set *X* and $a \in G(X)$.

Proposition 14[14]. Let *F* be a regular functor in *Comp*, *G* its regular subfunctor. Then

$$supp_{F(X)}(a) = supp_{G(X)}(a)$$

for any compact set *X* and $a \in G(X)$.

From propositions 9 and 14 it follows

Proposition 15[15]. Let *F* be a regular functor of degree $\le k$ in *Comp*, and let *G* be its regular subfunctor. Then $\deg G \le k$.

From the equality (7) we immediately obtain

Proposition 16[15]. Let *F* be a continuous functor in *Comp*, and *G* its continuous subfunctor. Then $G_k \subset F_k$ for any natural number k.

Proposition 17[15]. Let *F* be a regular functor in the category *Comp*, and let *G* be a regular subfunctor in *F*. Then

$$G_\beta(X) = G(\beta X) \bigcap F_\beta(X)$$

for any Tikhonov space *X*.

Proposition 18[11]. Let *F* - be a functor in the category *Comp*, and *G* be its subfunctor. Then G_β is a subfunctor of F_β.

About closed subfunctors and their sums

Definition 19[16]. Let *F* be a functor acting in the category $! \subset Top$, and let *G* be a subfunctor of *F*. A functor *G* is called a closed subfunctor of a functor *F* if for any $X \in O(C)$ the space $G(X)$ is a closed subspace of $F(X)$.

The next statement is trivial.

Proposition 20[16]. In the category *Comp*, each subfunctor is closed.

From Proposition 8 and Proposition 20 we get

Corollary 21[6]. For every continuous functor $F: Comp \to Comp$ and a natural number k, the functor F_k is a closed subfunctor of *F*.

Returning to the category *Tych*, for a Tikhonov space *X*, a continuous functor $F: Comp \to Comp$ and a positive integer *k*, we set

$$F_k(X) = \pi_{F,\beta,k}(C(k),X) \times F(k). \quad (10)$$

We also put

$$F_k(\varnothing) = F(\varnothing). \quad (11)$$

Let us now denote the restriction of $\pi_{F,\beta X,k}$ to

$C(k)\times F(k)$ by $\pi_{F,X,k}$. If $g:X\to Y$ is a continuous map, then

$$F(\beta g)(F_k(X))\subset F_k(Y),$$

taking into account the equality (9) to display $f=\beta g$. Therefore the setting

$$F_k(g)=F_k(\beta g)\,|\,F(X),$$

we obtain the mapping $F_k(g):F_k(X)\to F_k(Y)$.

Thus, we have defined a covariant functor

$$F_k:Tych\to Tych,$$

this extends the functor $F_k:Comp\to Comp$ to the category *Tych*.

The following statement follows from Proposition 9.

Proposition 22. [17]. If $F:Comp\to Comp$ is a continuous functor, then $F_k:Tych\to Tych$ is a subfunctor of the functor F_β , and

$$F_k(X)=F_\beta(X)\bigcap F_k(\beta X). \tag{12}$$

The equality (12) gives

Corollary 23[17]. For every continuous functor $F:Comp\to Comp$ and a natural number *k*, the functor $(F_k)_\beta$ is a closed subfunctor of F_β.

Comment 24[17]. Proposition 9 states that the definitions of the functor F_k [1] and Basmanov [3] coincide for any continuous functor *F* in $Comp$. However, we will use both definitions depending on the situation and denote them $F_{k,\beta}$ and $F_{k,b}$ respectively. As for the equality (10), we can assume that it defines the functor $(F_\beta)_{k,b}$. Thus, the equality (12) can be written in the form

$$(F_\beta)_{k,b}=(F_{k,b})_\beta=(F_{k,b})_\beta. \tag{13}$$

Proposition 25[17]. If *G* is a regular subfunctor of a regular functor *F* in *Comp*, then G_β is a closed subfunctor of F_β.

Definition 26. [11]. A functor *F* is said to be finitely open if the set $F_k(k+1)$ is open in $F(k+1)$ for any natural number *k*. The dual of this definition states that $F(k+1)\setminus F_k(k+1)$ is closed in $F(k+1)$.

Comment 27[17]. As an example of a finitely open functor, we can take any finitary functor (or finite functor) *F*, i.e. a functor *F* such that $F(k)$ is finite for any positive number *k*. In particular, the hyperspace functor exp[3] is finite and therefore finitely open.

Proposition 28[11,17]. If *F* is a regular finitely open functor, and *G* is its regular subfunctor, then *G* is finitely open.

Recall that an epimorphism $f:X\to Y$ is said to be inductively closed if there is a closed subset *A* in *X* such that $f(A)=Y$ and $f\,|_A$ is a closed mapping.

Definition 29[16-17]. A continuous functor $F:Comp\to Comp$ is said to be projectively inductively closed (p.i.c.) if the mapping $\pi_{F,X,k}$ is inductively closed for any Tikhonov space *X* and a positive integer *k*.

Theorem 30[16-17]. Every continuous finitely open functor *F* preserving the empty set and preimages is a p.i.c. functor.

Corollary 31[17]. Every finitary normal functor, in particular, the functor $\exp_k$, is a p.i.c.-functor.

Let $F:Tych\to Tych$ be a functor and $F^n\subset_{cl}F, n\in\omega$. We say that *F* is a union of F^n(i.e. $F=\bigcup_{n=0}^{\infty}F^n$), if $F(X)=\bigcup_{n=0}^{\infty}F^n(X)$ for any Tikhonov space *X*[17].

Definition 32[17]. The functor $F:Tych\to Tych$ is called σ -p.i.c. functor if $F=\bigcup_{n=0}^{\infty}(F^n)_\beta$, where each F^n is a p.i.c-functor of finite degree.

Example 33[17]. Taking into account Remark 27 and Theorem 30, the functor $\exp_\omega=\bigcup_{n=1}^{\infty}\exp_n$ is the functor σ -p.i.c.

The next statement is obvious.

Proposition 34[17]. If $F=\bigcup_{n=0}^{\infty}F^n$, where each F^n is a functor σ - p.i.c., then *F* is a functor σ -p.i.c.

Recall that $P_n:Comp\to Comp$ denotes the functor of Borel regular probability measures. This functor is normal [1,2,4]. According to Proposition 22 and Remark 24, we will denote $(P_k)_\beta$ by P_k.

Theorem 35. The functor P_k is a σ -p.i.c.-functor for any natural number k.

Comment 36. In fact, we have proven more: the functor P_k is the union of its normal finitely open subfunctors $(P_k^n)_\beta$.

We will say that the σ -p.i.c.-functor *F* has degree *k* if in the representation $F=\bigcup_{n=0}^{\infty}F^n$ (from Definition 32[17]) each functor F^n has degree $\le k$.

Proposition 37[17]. Let G_1 and G_2 be normal subfunctors of the normal functor $F:Comp\to Comp$ of finite degree $\le k$. Then their intersection $G=G_1\bigcap G_2$ is a normal functor of degree $\le k$.

Proposition 38. Let G_1,G_2 be normal subfunctors of the normal functor *F* in *Comp*. Then

$$(G_1)_\beta \bigcap (G_2)_\beta = (G_1 \bigcap G_2)_\beta$$

Theorem 39[11,17]. Let *F* be a normal subfunctor of P_k in *Comp*. Then F_β is a σ-p.i.c.-functor.

Definition 40[18]. A normal space *X* is said to be weakly countable-dimensional if *X* is the union of a countable family of closed subsets X_i such that $\dim X_i < \infty$ for each *i*.

The next two statements are trivial.

Proposition 41[18]. Every closed subspace of a weakly countable-dimensional space is a weakly countable-dimensional space.

Proposition 42[18]. Every normal space that is the union of a countable family of its closed weakly countable-dimensional subspaces is weakly countable-dimensional.

A completely regular space X is called a cirrus ($p-$space) [19] if there exists a countable family U_n of coverings of the space X by sets covered in its Stone–Cech extension $betaX$ such that $\bigcap_{n=1}^{\infty} 3_b(X, U_n) \subset X$ for each $x \in X$.

A space X is called a $\sum -$ space[19] if there exists a $\sigma -$discrete family N and covered C consisting of closed countably compact sets such that if $c \in C$ and $C \subset U$, then $C \subset F \subset U$ for some $F \in N$, where U is open in X.

A space that has a $\sigma -$locally finite network is called a $\sigma -$space[19]. those. X is a countable sum of locally finite families F_i, each of which is a network of the space X.

A $T_1 -$ space X is called a stratifiable (or lace) space (in short, a G-space) [19] if each open set $U \subset X$ can be associated with a sequence $\{U_n : n \in N\}$ open subsets in such a way that the following conditions are satisfied:

a) $\bar{U}_n \subset U$ for each $n \in N$; b) $\bigcup \{U_n : n \in N\} = U$; c) if $U \subset V$, then $U_n \subset V_n$ for all n.

Theorem 43[11]. Let *F*-p.i.c. be a functor of finite degree transforming finite sets into finite-dimensional compact sets, let *X* be a weakly countable-dimensional space, and let *X* belong to one from the following classes:

a) $\sum -$ paracompact spaces; b) *p*-paracompact spaces;

c) σ-paracompact spaces; d) stratified spaces;

e) metrizable spaces.

Then $F_\beta(X)$ is a weakly countable-dimensional space.

Corollary 44[11]. Let *F* be a normal finitary functor of finite degree, in particular, the functor $\exp_k$, *X* be a weakly countable-dimensional space, and let *X* belongs to one of the following classes:

a) $\sum -$ paracompact spaces; b) *p*-paracompact spaces;

c) σ-paracompact spaces; d) stratified spaces;

e) metrizable spaces.

Then $F_\beta(X)$ is a weakly countable-dimensional space.

Corollary 45[11].. Let *F* be a normal finitary functor of finite degree, in particular, the functor , *X* be a weakly countable-dimensional space, and let *X* belong to one of the following classes:

a) $\sum -$ paracompact spaces; b) *p*-paracompact spaces;

c) σ-paracompact spaces; d) stratified spaces;

e) metrizable spaces.

Then is a weakly countable-dimensional space.

Recall that for a normal functor $F : Comp \to Comp$, by F_ω they denote the σ-compact subfunctor F_β, defined as: $F_\omega(X) = \bigcup_{k=1}^{\infty} F_k(X)$ for any Tikhonov space *X*.

Thus, $F_\omega = \bigcup_{k=1}^{\infty} F_k$.

Theorem 46[17]. If *X* is a weakly countable-dimensional space that is *p*-paracompact (in particular, metrizable) or stratifiable M, then $P_\omega(X)$ is weakly countable-dimensional.

Proposition 41 and Theorem 46 give

Corollary 47[17]. If X is a weakly countable-dimensional space that is either *p*-paracompact (in particular, metrizable) or stratifiable, then $F(X)$ is a weakly countable-dimensional space for an arbitrary closed subfunctor $F \subseteq P_\omega$.

Applications

Theorem [2]. Every locally convex subfunctor F of a functor P_n preserves the property of the space to be an $A(N)R$-compact space and the property of a compact space to be a $Q -$manifold or a Hilbert brick.

Theorem [4]. Let X be a weakly countable-dimensional $A(N)R(\mathrm{M})$ space, and let $F -$ be a locally convex subfunctor of P_n. Then $F(X) \in A(N)R(\mathrm{M})$. In particular, $P_n(X) \in A(N)R(\mathrm{M})$.

Corollary [4]. Separable spaces $P_f(X)$, $P_f^c(X)$, $P_{f,n}(X)$ and $P_{f,n}^c(X)$ is metrizable by a complete metric if and only if X itself is separable and metrizable by a complete metric.

Corollary [8]. Let A be a $Z -$set of a $Q -$variety. Then the set A has a neighborhood homeomorphic to an open subset of the space Q.

From this corollary we can conclude that if a $Z-$ set A is connected in Q, then OA is a neighborhood of the cube Q and $\overline{OA}$; Q. those. any connected $Z-$ set A of a Hilbert cube Q has a closed neighborhood homeomorphic to the cube Q itself.

Lemma [5-6]. Let $A \subset Q$ be a $Z-$ set and $\overline{OA}$; Q. If $\overline{OA}$ $\varepsilon-$is close to id_Q, where $\overline{O(A)}$ is convex, then there is a retraction $r: Q \to Q$, $\varepsilon-$close to id_Q.

Has the following

Lemma [5-6]. For any compact set X, if the mapping $f: X \to P(X)$ $\varepsilon-$ is close to id_X, then $\psi \circ P(f): P(X) \to P(X)$ then $\varepsilon-$ close and $id_{P(X)}$, where $\psi: P^2(X) \to P(X)$ - retraction of the monad [6] .

Theorem [5-6]. Functors P_f, P_f^c, $P_{f,n}$, $P_{f,n}^c$, P_n, P_n^c and F - preserves the property of the layer in the mappings being $Q-$, $\sum -$, and $\ell_2^f -$ manifolds, where $F-$ are locally convex subfunctors of the functor P_n.

Corollary [5-6]. Functors P_f, P_f^c, $P_{f,n}$, $P_{f,n}^c$, P_n, P_n^c and F -preserves a) ℓ_2^f - varieties. In particular,

$$P_f(\ell_2^f) \approx P_f^c(\ell_2^f) \approx P_{f,n}(\ell_2^f) \approx$$
$$\approx P_{f,n}^c(\ell_2^f) \approx P_n(\ell_2^f) \approx P_n^c(\ell_2^f) \approx F(\ell_2^f) \approx \ell_2^f$$

; where $F-$ is a locally convex subfunctor of the functor P_n ;

b) Functors P_f, P_f^c, $P_{f,n}$, $P_{f,n}^c$, P_n, P_n^c and F - preserves $\sum -$ varieties. In particular,

$$P_f(\sum) \approx P_f^c(\sum) \approx P_{f,n}(\sum) \approx$$
$$\approx P_{f,n}^c(\sum) \approx P_n(\sum) \approx P_n^c(\sum) \approx F(\sum) \approx \sum$$

, where $F-$ locally convex subfunctors of the functor P_n.

Theorem [5]. Let $F-$be a continuous functor with finite supports preserving separable $A(N)R-$spaces. Then preserves $\sum -$varieties.

Theorem [5]. For functors $F = P_{f,n}^c$, $P_{f,n}$, P_f, P_f^c and locally convex subfunctors of the functor P_n the following pairs are homeomorphic:

$$\left(F(Q), F(S)\right); \ (Q, S)$$

$$\left(F(Q), BdQ\right); \ (Q, BdQ).$$

Note that the subspaces $P(X) \setminus P_n(X)$, $P(X) \setminus P_{f,n}(X)$ and $P(X) \setminus P_f(X)$ are a $Q-$ variety for any infinite compact set X and any $n \in N$. In the particular case $P(Q) \setminus P_n(Q)$, $P(Q) \setminus P_{f,n}(Q)$, $P(Q) \setminus P_f(Q) -$ is a $Q-$ variety. A subspace $P(X) \setminus P_\omega(X)$ is $G_\sigma -$set in $P(X)$.

For an infinite compact set X and a functor $\exp X$, the subspace $\exp_\nabla(X)$ and $\exp_{\nabla n}(X)$ are open in exp*X*, $exp_{nk}(X)$ is open in $\exp_n(X)$, $\exp_\omega(X)$ is a countable union of compacta in exp*X* , i.e. $\exp_\omega(X) - \sigma -$ compact. On the other hand, $\exp_\omega(X)$ is dense everywhere in exp*X*. a $\exp_{\nabla\omega}(X)$ is a F_δ -subspace of the space exp*X*, the subspace $F_{\omega n}(X)$ is an open set in $F_\omega(X)$.

The normal subfunctor of the functor P_n is the $n-$th symmetric degree functor SP^n. The component T_X of the natural transformation $T: SP^n \to P_n$ is defined by the equality

$$T_X\left[(x_1, x_2, ..., x_n)\right] = \frac{1}{n} \ ,\sum_{i=1}^{n} \delta_{x_i}$$

From the results of the main part it follows that the functor P_n and all its subfunctors are $\sigma - p.i.c.$ functors.

Consequently, the functor SP^n is also a $\sigma - p.i.c.$ functor, i.e. functors P_f, P_f^c, $P_{f,n}$, P_n^c, $P_{f,n}^c$, P_ω, P_ω^c and $F-$ are $\sigma - p.i.c.$ - functors, where F is a locally convex subfunctor of the functor P_n.

Note that the spaces ℓ_2^f and $\sum$ are everywhere dense subsets of the Hilbert cube Q and are preserved by the reduced functors F, then $\overline{F(\ell_2^f)}$ and $\overline{F(\sum)}$ is homeomorphic to the Hilbert cube Q i.e. $\overline{F(\ell_2^f)} \cong Q$ and $\overline{F(\sum)};\ Q$, where F is one of the following functors: locally convex functors, P_n, P_n^c, $P_{f,n}$, $P_{f,n}^c$, SP^n, P_f and P_f^c.

On the other hand, for these functors F, by Theorem 5.5[6]-5.8[6], $\overline{F(\ell_2^f)} = F(Q) = Q$ holds and $\overline{F(\sum)} = F(Q) = Q$.

Therefore, the following holds: $F(Q) \setminus F(\ell_2^f);\ S$

$$F(Q) \setminus F(\textstyle\sum);\ S.$$

For any compact set X we have:

$$SP^n(X) \subset SP^n(X) \subset P_n(X).$$

Theorem 1. For any infinite compact set X and for any $n \in N$, the subspace $P(X) \setminus SP^n(X)$ is homotopy dense in $P(X)$.

Proof. Let X be an infinite compact space. In this case $P(X);\ Q$. We fix $n \in N$. Take the measure $\mu_0 = m_1\delta_{x_1} + ... + m_k\delta_{x_k}$, where $k > n, , \sum_{i=1}^{k} m_i = 1$, $m_i > 0$, $m_i \neq m_j, \forall i \neq j$.

We construct the homotopy $h(\mu,t): P(X) \times [0,1] \to P(X)$ with a flat slope $h(\mu,t) = (1-t)\mu + t \cdot \mu_0$

For $t = 0$, $h(\mu,0) = (1-0)\mu + 0 \cdot \mu_0 = \mu$. those. $h(\mu,0) = id_{P(X)}$

For $t > 0$, $h(\mu,t) = (1-t)\mu + t \cdot \mu_0 \overline{\in} SP^n(X)$, so how $supph(\mu,t)$ consists of than $(n+1)-$ points. This means that the subspace $P(X) \setminus SP^n(X)$ is homotopy dense in $P(X)$.

Since the compact set $SP^n(X)$ is closed in $P(X)$, it follows that for an infinite compact set X and a natural number $n \in N$ the subspace $P(X) \setminus SP^n(X)$ is open in $P(X)$ i.e. $P(X) \setminus SP^n(X)$ are $Q-$ varieties. On the other hand, by virtue of Theorem [13], for any infinitely compact set X and for a natural number $n \in N$, the compact set $SP^n(X)$ is a $Z-$set in $P(X)$.

For an infinite compact set X we set $SP_\omega(X) = \bigcup_{n=1}^{\infty} SP^n(X)$. In this case we have

Theorem 2. For any infinite compact set X and any natural number $n \in N$, the subspace $P(X) \setminus SP^n(X)$ is a $Q-$ manifold and a subspace $SP_\omega^n(X) - \sigma - Z-$set in $P(X)$.

It is known that $SP^n(Q) = Q$ and $SP^n(X) \in A(N)R,$ if $X \in A(N)R$. Note that $SP_\omega(X) \subset P_\omega(X)$. For the functor SP^n and the Hilbert cube Q the following holds:

1) $SP_\Delta^n(Q)-$ is a $Q-$ manifold, since the subspace $SP_\Delta^n(Q)$ is open in $SP^n(Q)$;

2) $SP_{\Delta k}^n(Q) = SP^n(Q) \setminus SP^k(Q)$ are $Q-$ varieties, since for $k < n$ $SP^n(Q) \setminus SP^k(Q)$ is open in $SP^n(Q)$.

Theorem 3. For any infinite compact set X and for any $n \in N$, the subspace $P_{\omega n}(X)$ is homotopy dense in $P_\omega(X)$.

Proof. Let X be an infinite compact set and $n \in N$. Note that the space $P_\omega(X) \in AR$, since $P_\omega(X)$ is convex and locally convex. We construct the desired homotopy $h(\mu,t): P_\omega(X) \times [0,1] \to P_\omega(X)$ flat $h(\mu,t) = (1-t)\mu + t \cdot \mu_0$, where $\mu_0 \in P_{2n}(X)$. those. $|supp\mu_0| = 2n$.

If $t = 0$, then $h(\mu,0) = (1-0)\mu + 0 \cdot \mu_0 = \mu$. those. $h(\mu,0) = id_{P_\omega(X)}$.

If $t > 0$, then $h(\mu,t) = (1-t)\mu + t \cdot \mu_0 \in P_n(X)$, since $supph(\mu,t)$ contains more than $(n+1)$ distinct points. those. $h(\mu(0,1]) \in P_{\omega n}(X)$. ¦

From this Theorem 3 it follows that the compact set $P_n(X)$ is a $Z-$ set in $P_\omega(X)$. those. the space

$P_\omega(X)$ is a $\sigma - Z -$set. We noted that for any compact set X the following holds: $SP^n(X) \subset P_n(X)$.

From Theorem 3 it follows

Corollary 1. For any infinite compact set X and for any $n \in N$, the compact set $SP^n(X)$ is a $Z-$set in $P_\omega(X)$.

Corollary 2. For any infinite compact set X, the subspace $SP_\omega(X)$ is a $\sigma - Z -$set in $P_\omega(X)$.

Using the given properties of $\sigma - p.i.c. -$ functors and theorem [10] (Problem 16 § 1.2), the following can be easily proven.

Theorem 4. For $\sigma - p.i.c.$ functors $F : P_n$, $P_n^c, P_{f,n}, P_{f,n}^c, SP^n, SP_\omega, P_\omega,\ P_\omega^c$ and locally convex subfunctors P_n holds:

$$aF(\ell_2^f);\ \ Q;$$

$$aF(\sum);\ \ Q.$$

where $aF(X)$ is the one-point Alexandrov compactification of the space $F(X)$.

For a functor of symmetric degree SP^n we have:

1. For the segment $X = [0,1]$ the space $SP^n(X = I)$; σ^n is a standard simplex of dimension n; those. $SP^n([0,1]) \approx \sigma^n$;

2. If $X = S^2$ is a sphere in R^3, then $SP^n(S^2)$; $CP^n isa$ projective (complex) space dimensions n;

3. If $X \quad S' -$ is a circle in R^2, then $SP^2(S^1)$; $\exp_2 S^1$; $M_2 -$ Miyobius sheet in R^3;

4. If $X = S' -$ circle in R^2, then $\exp_3 S'$; $S^3 -$ sphere of dimension 3 in R^4, where $\exp_3 S'$ is a set consisting of no more than

5. If X is a circle in R^2, then $\exp_c S'$; B^2 is a closed circle in R^2, where $\exp_c S'$ consists of connected continua of the circle $S' \subset R^2$.

REFERENCES

- Федорчук В.В. Ковариантные функторы в категории компактов, абсолютные ретракты и *Q*-многообразия Успехи матем. наук. Москва, 1981. Т.36. Вып.3.с. 177-195.
- Федорчук В.В. Вероятностные меры в топологии. Успехи матем. наук. Москва, 1991. Т.46. Вып.1 277. с.41-80.
- Щепин Е.В. Функторы и несчетные степени компактов Успехи мат. наук. Москва, 1981. Т.36. 3. с. 3-62.
- Жураев Т.Ф. Некоторые геометрические свойства подфункторов функтора *P* вероятностных мер. Москва, МГУ, 1989. 60 с., Деп. в ВИНИТИ АН СССР 05.07.89. 4471-В89.
- Жураев Т.Ф. Некоторые геометрические свойства функтора вероятностных мер и его подфункторов: Дис. канд. физ.-мат.-М: МГУ, 1989. 90 с.
- Заричный М.М. Абсолютные экстензоры и геометрия умножения монад в категории компактов. Мат. сб. 1991, 182, 9, с. 1261-1280.
- Богатый С.А., Федорчук В.В. Теория ретрактов и бесконечномерные многообразия Итоги науки и техники. Алгебра. Топология. Геометрия, 1986. Т.24. с. 195-270.
- Чепмэн Т. Лекции о -многообразиях. Москва: Мир, 1981.-170 с.
- Bessaga C., Pelczunski A. Selected topics in infinity dimensional topology. PWN, Warsaw, 1975, 58, 353 p.
- Banakh, T. Radul, M. Zarichnyi Absorbing sets in infinite – dimensional Manifolds, Math Studies Monogh. Ser. V.1. VNTL Publishers, 1996, 232 p.
- Жураев Т.Ф., Геометрические и топологические свойства пространств, являющихся значениями некоторых ковариантных функторов. Ташкент.2022, Доктор.-дис. Мат. инс. Романовского АН Узбекистана, 198 с.
- Banakh T.O. Topology of probability measure spaces. I. Matem Stud. Proc. Lwow Math. Soc. 5 , 1995, p. 65-87.
- Ayupov Sh. A. Zhuraev T. F. On projectively inductively closed subfunctors of the functor P of probability measures. Journal of Mathematical Sciences. 2020. Vol. 245(3). pp. 382-389.
- Zhuraev T.F. On projectively quotrient functors. Comment. Math. Univ. Corolinae, 2001. V.42. 3. p.p. 561-573.
- Zhuraev T.F. On paracompact spaces and projectively inductively closed functors. Applied General Topology. 2002. Vol.3(1). pp. 33-44.
- ZhuraevT.F. On paracompact spaces, projectively inductively closed functors, and dimension. Mathematica Aeterna, International Journal for Pure and Applied Mathematics. Bulgariya. 2015. 6. pp. 175-189.
- Zhuraev T.F. On dimension and $\sigma - p.i.c.$ functors. Mathematica Aeterna, International Journal for Pure and Applied Mathematics. Bulgariya 2014, 6, pp.577-596.
- Федорчук В.В. Слабосчетномерные пространства Усп. мат. Наук, 2007, Т. 62, 2 (374), с.109-164.
- Энгелькинг Р. Общая топология. Москва. Мир, 1986. 752 с.